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A REMARK ON PROPER SEQUENCES OF MODULES*

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ABSTRACT. A bound for the depth of a quotient of the symmetric algebra, $S(E)$, of a finitely generated module E, over a C.M. ring by an ideal of $S(E)$ generated by a subsequence of $x_1,\ldots,x_n$ is obtained in the case when E satisfies the sliding depth condition, with maximal irrelevant ideal generated by a proper sequence $x_1,\ldots,x_n$ in E.

Introduction

Let R be a commutative noetherian C.M. local ring of dimension d, and let E be a finitely generated R-module of rank e.

We denote with $Sym_R(E)$ or with $S(E)$, the symmetric algebra of E over R, that is the graded algebra over R

$$S(E) = \bigoplus_{t\geq 0} Sym_t(E),$$

with $S_+ = \mathbf{x} = (x_1, \ldots, x_n)$ the graded maximal irrelevant ideal of $S(E)$ and with $S = R[T_1, \ldots, T_n]$ the polynomial ring in n variables.

When $\mathbf{x} = \{x_1, \ldots, x_n\}$ is a proper sequence in E, $\mathbf{x}_i = \{x_1, \ldots, x_i\}$ is a subsequence of $\mathbf{x}$, $i = 1, \ldots, n$, the complex of homology modules of the Koszul complex on the elements $\mathbf{x}_i$ is exact and give us informations on the quotient ring $S(E)/(x_{i+1}, \ldots, x_n)$.

It is possible to obtain bounds on the depth of $S(E)/(x_{i+1}, \ldots, x_n)$ knowing bounds on the depth of the ith components of the homology modules of the Koszul complex. For example, the condition SD_k on modules is able to give us such conditions, introduced in [5].

In this article we obtain a bound for the depth of the quotient ring $S(E)$ by ideals generated by proper sequences of 1-forms of the maximal irrelevant ideal of $S(E)$, under weaker hypothesis given in theorem 2 of [9], where we used approximation complex $\mathcal{Z}(E)$ of the module E.

We obtain the following:

Theorem 1. *Let (R,m) be a C.M. local ring of dimension d. Let E a f.g. R-module that satisfies SD_k and $x_1,\dots,x_n$ a proper sequence of E.*
Then $\operatorname{depth} S(E)/(\boldsymbol{x}_{n-i+1,n}) \geq d-i+k$, $i=0,\dots,n$.

Our result generalizes to the case of a module the result in [6], where the problem is studied in the case of an ideal $I \subset R$ generated by proper sequences, obtaining bounds on the depth of the quotient of the ring R by I.

1. Preliminaries

Let R be a commutative noetherian C.M. local ring of dimension d, and let E be a finitely generated R-module of rank e.

We denote with $Sym_R(E)$ or with $S(E)$, the symmetric algebra of E over R, that is the graded algebra over R:

$$S(E) = \bigoplus_{t\geq 0} Sym_t(E)$$

and with S_+ the maximal irrelevant ideal of $S(E)$

$$S_+ = \bigoplus_{t>0} Sym_t(E).$$

Let $S_+ = (x_1,\dots,x_n) = (\mathbf{x})$. We can consider the Koszul complex on the generating set $\{x_1,\dots,x_n\}$ of S_+, that is a graded complex.

In particular in degree $t>0$ we have

$$0 \to \bigwedge^n R^n \otimes S_{t-n}(E) \xrightarrow{d_n} \bigwedge^{n-1} R^n \otimes S_{t-n+1}(E) \xrightarrow{d_{n-1}} \cdots$$
$$\cdots \bigwedge^2 R^n \otimes S_{t-2}(E) \xrightarrow{d_2} R^n \otimes S_{t-1}(E) \xrightarrow{d_1} S_{t-1}(E) \to 0$$

with differential d_p

$$d_p(e_{i_1}\wedge\cdots\wedge e_{i_p}\otimes f(\mathbf{x})) = \sum_{j=1}^{p}(-1)^{p-j} e_{i_1}\wedge\cdots\wedge e_{i_{j-1}}\wedge e_{i_{j+1}}\wedge\cdots\wedge e_{i_p}\otimes x_{i_j} f(\mathbf{x}).$$

We denote with $H_i(\mathbf{x};S(E))_j$, $j \geq i$, the $j-$th graded component of the Koszul homology $H_i(\mathbf{x};S(E))$.

It results:

$$H_i(\mathbf{x};S(E)) = \bigoplus_{j\geq i} H_i(\mathbf{x};S(E))_j$$

Now, is possible to define the complex:

$$0 \to H_n(\mathbf{x},S(E))_n \otimes S[-n] \to \cdots \to H_1(\mathbf{x},S(E))_1 \otimes S[-1] \to S$$

and in general the complexes associated to the subsequences
$\mathbf{x}_i = \{x_1,\dots,x_i\}$

$$0 \to H_i(\mathbf{x}_i,S(E))_i \otimes S[-i] \to \cdots \to H_1(\mathbf{x}_i,S(E))_1 \otimes S[-1] \to S$$

Definition 1. [1] *Let R be a graded ring. A graded ideal $\mathfrak{m}$ of R is called *maximal, if every graded ideal that contains $\mathfrak{m}$ properly, equals R. The ring R is called *local, if it has a unique *maximal ideal $\mathfrak{m}$. A *local ring R with *maximal ideal $\mathfrak{m}$ will be denoted by $(R, \mathfrak{m})$.*

Example 1. *Let $R = \oplus_{i\geq 0} R_i$ be a graded ring for which R_0 is a local ring with maximal ideal $\mathfrak{m}_0$, and call R_+ the ideal $\oplus_{i>0} R_i$. Then R is a *local ring with *maximal ideal $\mathfrak{m} = \mathfrak{m}_0 \oplus R_+$.*

Remark 1. *Let M, N graded R-module. We can denote by $\mathrm{Hom}_i(M, N)$ the module of homogeneous homomorphisms of degree i.*

We can define $^*\mathrm{Hom}(M, N) = \bigoplus \mathrm{Hom}_i(M, N)$ as the submodule of homogeneous homomorphisms of $\mathrm{Hom}(M, N)$.

It is known that $^*\mathrm{Hom}(M, N) = \mathrm{Hom}(M, N)$ when M is finitely generated ([1], chapter 1.5).

The same remark is possible to give for the i-th right derived functor

$$* \mathrm{Ext}^i \text{ of} * \mathrm{Hom}(\_, N),$$

that is $^*\mathrm{Ext}^i(M, N) = \mathrm{Ext}^i(M, N)$, when M is finitely generated ([1], chapter 1.5).

Definition 2. *Let M be a graded module on a *local ring $(R, \mathfrak{m})$. A sequence $\boldsymbol{x} = x_1, \ldots, x_n$ of graded elements in R is an M-regular sequence if the following conditions are satisfied:*
1) x_i is an $M/(x_1, \ldots, x_{i-1})M$-regular element for $i = 1, \ldots, n$;
2) $(\boldsymbol{x})M \neq M$.

Now is possible to define the depth for a *local ring

Definition 3. *Let M be a graded module on a *local ring $(R, \mathfrak{m})$. The depth of M w.r.t. $\mathfrak{m}$, denoted with*

$$\mathrm{depth}_{\mathfrak{m}} M$$

is the length of the maximal M-regular sequence of graded elements contained in $\mathfrak{m}$, or equivalently

$$\min\{i : {}^*\mathrm{Ext}^i_R(R/\mathfrak{m}, M) \neq 0\}.$$

Lemma 1. *Let $(R, \mathfrak{m})$ be a *local ring and*

$$0 \to U \to M \to N \to 0$$

an exact sequence of graded and finitely generated R-modules. Then
1) $\mathrm{depth}_{\mathfrak{m}} M \geq \min\{\mathrm{depth}_{\mathfrak{m}} U, \mathrm{depth}_{\mathfrak{m}} N\}$;
2) $\mathrm{depth}_{\mathfrak{m}} U \geq \min\{\mathrm{depth}_{\mathfrak{m}} M, \mathrm{depth}_{\mathfrak{m}} N + 1\}$;
3) $\mathrm{depth}_{\mathfrak{m}} N \geq \min\{\mathrm{depth}_{\mathfrak{m}} U - 1, \mathrm{depth}_{\mathfrak{m}} M\}$.

Proof: The assertion is proved using proposition 1.2.9 of [1] and remark 1.

Definition 4. [10] *Let $\boldsymbol{x} = x_1, \ldots, x_n$ be a sequence of 1-form generating the maximal irrelevant ideal S_+ of $S(E)$. Then $\boldsymbol{x}$ is called a proper sequence in E if:*

$$x_{i+1} Z_j(x_1, \ldots, x_i; S(E))_j / B_j(x_1, \ldots, x_i; S(E))_{j+1} = 0 \qquad 0 \leq i \leq n-1, j > 0.$$

where $Z_j(x_1,\dots,x_i;S(E))_j$ and $B_j(x_1,\dots,x_i;S(E))_j$ are respectively the the $j-th$ graded component of the cycles and boundaries of the Koszul complex.

Now we recall the sliding depth condition SD_k over finitely generated modules (see [5]).

Definition 5. *Let $(R,\mathfrak{m}_0)$ be a C.M. local ring of dimension d. Let E be a finitely generated R-module, $S_+ = (x_1,\dots,x_n) = (\boldsymbol{x})$ the maximal irrelevant ideal of the symmetric algebra $S_R(E)$.*

We say that E satisfies the sliding depth condition SD_k, with k integer, if $\forall i \geq 0$:

$$\operatorname{depth}_{\mathfrak{m}_0} H_i(\boldsymbol{x},S(E))_i \geq d-n+i+k, \qquad 0 \leq i \leq n-k.$$

If $k = rank(E)$ we shall say that E satisfies the sliding depth condition SD.

Remark 2. *In particular if E satisfies the sliding depth condition SD_k, with $S_+ = (x_1,\dots,x_n) = (\boldsymbol{x})$ the maximal irrelevant ideal of the symmetric algebra $S_R(E)$, then for every subsequence $\boldsymbol{x}_j = \{x_1,\dots,x_j\}$ of $\boldsymbol{x}$ we have*

$$\operatorname{depth}_{\mathfrak{m}_0} H_i(\boldsymbol{x}_j,S(E))_i \geq d-j+i+k, \qquad 0 \leq i \leq n-k.$$

2. Main result

Let $(R,\mathfrak{m}_0)$ be a local ring and E a f.g. R-module. In this section we look at the symmetric algebra $S_R(E)$ of a module E, and a proper sequence in E, $\mathbf{x} = \{x_1,\dots,x_n\}$, generating the maximal irrelevant ideal S_+, and we call $S = R[T_1,\dots,T_n]$ the polynomial ring with coefficients in R with the natural grading.

Lemma 2. *Let $(R,\mathfrak{m}_0)$ be a C.M. local ring of dimension d. Let $\boldsymbol{x} = \{x_1,\dots,x_n\}$ a proper sequence of the module E.*
We call $\boldsymbol{x}_i = \{x_1,\dots,x_i\}$ and $\boldsymbol{x}_{i,n} = \{x_i,x_{i+1},\dots,x_n\}$ subsequence of $\boldsymbol{x}$, $H_j(\boldsymbol{x}_i)_l = H_j(\boldsymbol{x}_i;S(E))_l$. The following sequences are exact:
1)
$0 \to H_{j+1}(\boldsymbol{x}_i)_{j+1} \otimes S[-j-1] \to H_{j+1}(\boldsymbol{x}_{i+1})_{j+1} \otimes S[-j-1] \to H_{j+1}(\boldsymbol{x}_i)_{j+1}[-1] \otimes S[-j-1] \to 0$
2)
$0 \to Q^{(i)} \to S(E)/(\boldsymbol{x}_{i+1,n}) \to S(E)/(\boldsymbol{x}_{i,n}) \to 0$, *with* $Q^{(i)} = (\boldsymbol{x}_{i,n})/(\boldsymbol{x}_{i+1,n})$
3)
$0 \to M^{(i)} \to S(E)/(\boldsymbol{x}_{i+1,n}) \xrightarrow{x_i} Q^{(i)} \to 0$, *with* $M^{(i)} = ((\boldsymbol{x}_{i+1,n}) : x_i)/(\boldsymbol{x}_{i+1,n})$
4) $0 \to H_1(\boldsymbol{x}_i)_1 \otimes S[-1] \to H_1(\boldsymbol{x}_{i+1})_1 \otimes S[-1] \to M^{(i)} \to 0$, *where $M^{(i)}$ is seen as an S-module.*

Proof: Let $x = \{x_1,\dots,x_n\}$ a proper sequence of E. Then $\forall j > 1$
$0 \to H_j(\mathbf{x}_i)_j \otimes S[-j] \to H_j(\mathbf{x}_{i+1})_j \otimes S[-j] \to H_{j-1}(\mathbf{x}_i)_j \otimes S[-j] \to 0$
the sequences of S-modules are exact.
The tail of this homology sequence is
$0 \to H_1(\mathbf{x}_i)_1 \otimes S[-1] \to H_1(\mathbf{x}_{i+1})_1 \otimes S[-1] \to S(E)/(\mathbf{x}_{i+1,n}) \xrightarrow{x_i} S(E)/(\mathbf{x}_{i+1,n}) \to S(E)/(\mathbf{x}_{i,n}) \to 0.$

We can observe that kernel of the omomorphism

$$S(E)/(\mathbf{x}_{i+1,n}) \to S(E)/(\mathbf{x}_{i,n}) \to 0$$

is $Q^{(i)} = (\mathbf{x}_{i,n})/(\mathbf{x}_{i+1,n})$.

Now, considering the morphism

$$S(E)/(\mathbf{x}_{i+1,n}) \xrightarrow{x_i} Q^{(i)} \to 0,$$

we obtain the kernel that is $M^{(i)} = ((\mathbf{x}_{i+1,n}) : x_i)/(\mathbf{x}_{i+1,n})$,

and so we can break the long sequence (*) into the shorter sequence 2), 3) and 4).

Let $\mathbf{T} = \{T_1, \dots, T_n\}$, from now on, the depth of each S-modules, and therefore also $S(E)$-modules, is calculated with respect to the *maximal ideal

$$\mathfrak{m} = \mathfrak{m}_0 \oplus (\mathbf{T}).$$

Remark 3. *Let* $\mathfrak{m} = \mathfrak{m}_0 \oplus (\boldsymbol{T})$ *and* $\mathfrak{m}' = \mathfrak{m}_0 \oplus S_+$. *Then*

$$\operatorname{depth}_{\mathfrak{m}} S(E) = \operatorname{depth}_{\mathfrak{m}'} S(E),$$

where depth *w.r.t.* $\mathfrak{m}$ *is calculated considering* $S(E)$ *as an* S*-module, while* depth *w.r.t.* $\mathfrak{m}'$ *is calculated considering* $S(E)$ *as an* $S(E)$*-module.*

Remark 4. *Let* $(R, \mathfrak{m}_0)$ *be local ring and* E *a f.g.* R*-module.*

If $\boldsymbol{x} = \{x_1, \dots, x_n\}$ *is a proper sequence of* E, $\boldsymbol{x}_i = \{x_1, \dots, x_i\}$ *a subsequence of* $\boldsymbol{x}$, *then the following complexes of* S*-modules are exact:*

$$0 \to H_n(\boldsymbol{x}, S(E))_n \otimes S[-n] \to \cdots \to H_1(\boldsymbol{x}, S(E))_1 \otimes S[-1] \to S \to S(E);$$

$$0 \to H_i(\boldsymbol{x}_i, S(E))_i \otimes S[-i] \to \cdots$$
$$\to H_1(\boldsymbol{x}_i, S(E))_1 \otimes S[-1] \to S \to S(E)/(x_{i+1}, \dots, x_n).$$

Theorem 1. *Let* $(R, \mathfrak{m}_0)$ *be a C.M. local ring of dimension* d *and* E *a f.g.* R*-module. Suppose that:*

1) E *satisfies* SD_k*;*

2) $x_1, \dots, x_n$ *is a proper sequence of* E.

Then $\operatorname{depth}_{\mathfrak{m}} S(E)/(\boldsymbol{x}_{n-i+1,n}) \geq d - i + k$, $i = 0, \dots, n$.

Proof:

We suppose that the assertion is true for $j = n - i + 1$ and by contradiction let

$$\operatorname{depth}_{\mathfrak{m}} S(E)/(\mathbf{x}_{j-1,n}) = l < d - i + k - 1 \qquad (*).$$

Consider the exact sequence

$$0 \to H_1(\mathbf{x}_{j-1}; S(E))_1 \otimes S[-1] \to H_1(\mathbf{x}_j; S(E))_1 \otimes S[-1] \to M^{(j-1)} \to 0$$

By lemma 1, we have

$$\operatorname{depth}_{\mathfrak{m}} M^{(j-1)} \geq$$
$$min\{\operatorname{depth}_{\mathfrak{m}} H_1(\mathbf{x}_{j-1}; S(E))_1 \otimes S[-1] - 1, \operatorname{depth}_{\mathfrak{m}} H_1(\mathbf{x}_j; S(E))_1 \otimes S[-1]\}$$
$$= d + k - j + 1 + n > l$$

From the sequence of $S(E)$-modules (and therefore S-modules)

$$0 \to M^{(j-1)} \to S(E)/(\mathbf{x}_{j,n}) \to Q^{(j-1)} \to 0$$

we obtain the long exact sequence:

$$\cdots \to^* \operatorname{Ext}^{l-1}(k, M_{j-1}) \to^* \operatorname{Ext}^{l}(k, S(E)/(\mathbf{x}_{j,n})) \to$$
$$^* \operatorname{Ext}^{l}(k, Q^{(j-1)}) \to^* \operatorname{Ext}^{l+1}(k, M^{(j-1)}) \to \cdots$$

where $k \cong R/\mathfrak{m}_0$.

But $^* \operatorname{Ext}^{l-1}(k, M^{(j-1)}) = 0$, since $\operatorname{depth} M^{(j-1)} > l$, it follows that the map

$$\alpha :^* \operatorname{Ext}^{l}(k, S(E)/(\mathbf{x}_{j,n})) \to^* \operatorname{Ext}^{l}(k, Q^{(j-1)})$$

is injective.

In the same way, from the exact sequence of $S(E)$-modules

$$0 \to Q^{(j-1)} \to S(E)/(\mathbf{x}_{j,n}) \to S(E)/(\mathbf{x}_{j-1,x_n}) \to 0$$

we obtain

$$\cdots \to^* \operatorname{Ext}^{l-1}(k, S(E)/(\mathbf{x}_{j-1,n})) \to^* \operatorname{Ext}^{l}(k, Q^{(j-1)}) \to$$
$$^* \operatorname{Ext}^{l}(k, S(E)/(\mathbf{x}_{j,n})) \to^* \operatorname{Ext}^{l}(k, S(E)/(\mathbf{x}_{j-1,n})) \to \cdots$$

But $^* \operatorname{Ext}^{l-1}(k, S(E)/(\mathbf{x}_{j-1,n})) = 0$ by hypothesis (*),
so the map

$$\beta :^* \operatorname{Ext}^{l}(k, Q^{(j-1)}) \to^* \operatorname{Ext}^{l}(k, S(E)/(\mathbf{x}_{j,n}))$$

is injective, too. Therefore the composite $\beta\alpha$ is injective.

But this gives us a contradiction, since $\beta\alpha$ is induced by the multiplication by x_{j-1}, and it is the null mapping, since $\{x_1, \dots, x_n\}$ is a proper sequence.

Example 2. *Let $R = k[x_1, \dots, x_d]$, $I_1 = (f_1)$ and $I_2 = (f_2, f_3)$ ideals of R with f_1, f_2, f_3 monomials and $E = I_1 \oplus I_2$.*

Since E has rank 2

$$H_i(\boldsymbol{x}, S(E))_i = 0$$

for $i > 1$ (see [5]).

If $i = 1$, $H_1(\boldsymbol{x}, S(E))_1$ is the first syzygy module of E and by easy calculations we have

$$H_1(\boldsymbol{x}, S(E))_1 = Rf,$$

$f = (0, f_3/GCD[f_2, f_3], f_2/GCD[f_2, f_3]) \in R^3$, and in particular $Rf \cong R$. Therefore E satisfies SD_2.

The sequence f_1, f_2, f_3 is a strong s-sequence in the sense of [4], so $y_1, y_2, y_3 \in S(E) \cong R[y_1, y_2, y_3] \cong R[Y_1, Y_2, Y_3]/J$ is a d-sequence (see [4]) and this implies that the sequence is a proper sequence, too.

We have that

(1) $depth_{\mathfrak{m}} S(E) \geq d + 2$;
(2) $depth_{\mathfrak{m}} S(E)/(y_3) \geq d - 1 + 2$;
(3) $depth_{\mathfrak{m}} S(E)/(y_2, y_3) \geq d - 2 + 2$;
(4) $depth_{\mathfrak{m}} S(E)/(y_1, y_2, y_3) \geq d - 3 + 2$.

then the assertion of theorem 1 is satisfied.

Remark 5. *The computation of the depth was performed using CoCoA (see* [2]*) a computer algebra system entirely devoted to computing in polynomial rings. Other examples have been verified.*

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